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A diffusion wave equation with two fractional derivatives of different order

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Abstract
We analyse a diffusion wave equation with two fractional derivatives of different order on bounded and unbounded spatial domains. Thus, our model represents a generalized telegraph equation. Solutions to signalling and Cauchy problems in terms of a series and integral representation are given. Classical wave and heat conduction equations are obtained as limiting cases.

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1. Introduction

It is known that the classical heat equation is of parabolic type and, as a consequence, has physically unacceptable property that heat impulses propagate at an infinite speed in a bar in which heat is introduced at an arbitrary point. This property may be removed by assuming, as Cattaneo did in [1], that the thermal flux depends on the temperature gradient *and* on a special 'mixed' functional (see [2]). As a matter of fact, Cattaneo's equation for the heat flux may be expressed as an integral of the history of the temperature gradient. This interpretation is given in [3]. When these constitutive assumptions are made, the resulting partial differential equation, describing one-dimensional heat conduction, becomes

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \mathcal{D} \frac{\partial^2 T}{\partial x^2}, \qquad x \in (0, l), \qquad t > 0, \tag{1.1}$$

where T is the temperature, t is the time, x is the spatial coordinate, $\tau > 0$ is the relaxation time and $\mathcal{D} > 0$ is the thermal conductivity (also called thermal diffusivity). In what follows we shall allow for the spatial domain to be finite ($l < \infty$) and infinite ($l = \infty$). Equation (1.1) is known as a telegraph equation. In the special case when the relaxation time is zero, equation (1.1) becomes the classical heat conduction equation, i.e.,

$$\frac{\partial T}{\partial t} = \mathcal{D} \frac{\partial^2 T}{\partial x^2}, \qquad x \in (0, l), \qquad t > 0. \tag{1.2}$$

The generalization of (1.2) to a fractional order equation was proposed by Mainardi and his collaborators [4–7] and in his seminal papers [8] (see also [12]) as

$$\frac{\partial^\alpha T}{\partial t^\alpha} = \mathcal{D}\frac{\partial^2 T}{\partial x^2}, \tag{1.3}$$

with $1 < \alpha < 2$. We refer to [4] for a detailed account of the history of equation (1.3). In (1.3), we use $\frac{\partial^\alpha T}{\partial t^\alpha}$ to denote the Riemann–Liouville fractional derivative of order α with respect to time t of the function $T(x, t)$. For any $\beta \in \mathbb{R}$ and a function $f(t)$, supported by $[0, \infty)$, so that its $(m-1)$ th derivative is absolutely continuous, the fractional derivative of the order β is defined as

$$\frac{\mathrm{d}^\beta f}{\mathrm{d}t^\beta} = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\beta)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\beta+1-m}}\,\mathrm{d}\tau\right], \tag{1.4}$$

where Γ is the Euler gamma function and m is an integer satisfying $m - 1 \leqslant \beta < m$.

We also refer to papers [9–11] for the distributed order diffusion equations.

We shall treat in this paper the following fractional generalization of the telegraph equation (1.1):

$$\tau\frac{\partial^\alpha T}{\partial t^\alpha} + \frac{\partial^\beta T}{\partial t^\beta} = \mathcal{D}\frac{\partial^2 T}{\partial x^2}, \qquad x \in (0, l), \qquad t > 0. \tag{1.5}$$

In general, we shall take $0 < \beta \leqslant \alpha \leqslant 2$. Thus, we shall be able to recover both classical heat conduction equation $(\tau = 0, \beta = 1)$, the telegraph equation $(\alpha = 2, \beta = 1)$ and the classical wave equation $(\alpha = \beta = 2)$.

We shall treat two problems for (1.5) with $l = \infty$

(1) *Signalling* problem

$$\begin{aligned} &T(x, 0^+) = 0, && \frac{\partial T(x, 0^+)}{\partial t} = 0, && x \in (0, \infty), \\ &\lim_{x\to 0} T(x, t) = h(t), && \lim_{x\to\infty} T(x, t) = 0, && t > 0. \end{aligned} \tag{1.6}$$

(2) *Cauchy* problem

$$\begin{aligned} &T(0, t) = g(t), && T(x, 0^+) = 0, && \lim_{x\to\infty} T(x, t) = 0, \\ &x \in (0, \infty), && t > 0. \end{aligned} \tag{1.7}$$

In (1.7) and (1.6), we use $g(x)$ and $h(t)$ to denote two given sufficiently well-behaving functions of space and time variable, respectively. We note that for the case when $g(x) = \delta(x)$ and $h(t) = \delta(t)$ where δ is the Dirac distribution, the solutions to (1.7) and (1.6) determine Green functions, denoted by $\mathcal{G}_c(x, t)$ and $\mathcal{G}_s(x, t)$, respectively. With Green functions known, the solution to problem (1.7) and (1.6) with arbitrary $g(x)$ and $h(t)$ reads (see also [7])

$$T(x, t) = \int_0^\infty \mathcal{G}_c(x - \xi, t) g(\xi)\,\mathrm{d}\xi, \qquad T(x, t) = \int_0^\infty \mathcal{G}_s(x, t - \xi) h(\xi)\,\mathrm{d}\xi. \tag{1.8}$$

Also, we note that for the case when $1 < \alpha \leqslant 2$, we must add the first time derivative of $T(x, t)$ at the initial time, i.e., $\frac{\partial T}{\partial t}(x, 0^+) = f(x)$ to problems (1.5) and (1.7). In order to be able to compare results for the case $\alpha < 1$ with the case $1 \leqslant \alpha$, we set $f(x) = 0$.

For the case $l < \infty$, we shall treat (1.5) with the assumptions

$$T(0, t) = T(l, t) = 0. \tag{1.9}$$

2. Properties and solutions of problems for $0 < \alpha, \beta \leqslant 1$

2.1. The maximum principle

First, we analyse the solutions to (1.5) when $0 \leqslant \beta \leqslant \alpha \leqslant 1$. Let D be a rectangle in the (x, t) plane $D = \{(x, t)|0 < x < l, 0 < t < \widehat{t}\}$ with $\widehat{t} > 0$ given. We denote by $\overline{D}$ the closure of D and by γ the part of the boundary of D consisting of the sides $t = 0$, $x = 0$ and $x = l$. This is a closed set.

We state the following theorem that is an analogue of the classical maximum principle. Recall, $\tau > 0, \mathcal{D} > 0$.

Theorem 1. *Suppose that $T(x, t)$ is given and*

- *satisfies*

$$\tau \frac{\partial^\alpha T}{\partial t^\alpha} + \frac{\partial^\beta T}{\partial t^\beta} = \mathcal{D} \frac{\partial^2 T}{\partial x^2}, \qquad (x, t) \in D; \tag{2.1}$$

- *is continuous in both variables for* $(x, t) \in \overline{D}$ and $T(x, \cdot)$ *is Hölder continuous with the exponent* $h > \alpha$ for $t \in [0, \widehat{t}\,]$;
- *satisfies the following initial and boundary conditions:*

$$T(x, 0^+) = g(x), \qquad T(0, t) = f_1(t), \qquad T(l, t) = f_2(t); \tag{2.2}$$

- *the functions* $g(x)$, $f_1(t)$ and $f_2(t)$ are *continuous and satisfy*

$$g(0) = f_1(0), \qquad g(l) = f_2(0). \tag{2.3}$$

Then the maximum and minimum values of $T(x, t)$ over a region $\overline{D} = \{(x, t)|0 \leqslant x \leqslant l, 0 \leqslant t \leqslant \widehat{t}\}$ are assumed on γ.

Proof. It is enough to prove the theorem for the maximum. Since $T(x, t)$ is continuous, it reaches its maximum in $\overline{D}$. If $T(x, t)$ reaches its maximum at the point (x^*, t^*) with $0 < x^* < l, 0 < t^* < \widehat{t}$, then at this point we must have

$$\frac{\partial^2 T}{\partial x^2}(x^*, t^*) \leqslant 0. \tag{2.4}$$

Also, from the generalized Fermat's theorem for stationary points [[13], p 103] at the point t^*, where $T(x^*, \cdot)$ reaches maximum, the inequality

$$\frac{\partial^\alpha T}{\partial t^\alpha}(x^*, t^*) \geqslant 0 \tag{2.5}$$

holds for each $\alpha \in (0, 1)$. If at least in one of (2.4) or (2.5) we have the strict inequality, then (1.5) is not satisfied at (x^*, t^*) since the relaxation time τ is positive. In the case when $T = \text{const}$, we have $\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{\text{const}}{\Gamma(1-\alpha)} t^{-\alpha}$ and $\frac{\partial^2 T}{\partial x^2} = 0$, again leading to contradiction that (2.1) is satisfied in D.

So, suppose that in (2.4) and (2.5), we have the equality.

Suppose further that the maximum of $T(x, t)$ on the boundary γ is M while the maximum of $T(x, t)$ in D is attained at the point (x^*, t^*) and that $T(x^*, t^*) = M + \varepsilon$, $\varepsilon > 0$. We will show that this leads to a contradiction.

Consider a function $U(x, t) = T(x, t) - kt^\alpha$, $(x, t) \in \overline{D}$. Choose $k > 0$ so that $k < \frac{\varepsilon}{2(\widehat{t})^\alpha}$. Then, $\sup_{(x,t)\in D} U(x, t) \geqslant M + \varepsilon/2$ and $\sup_{(x,t)\in\gamma} U(x, t) \leqslant M$. Since U is continuous in $\overline{D}$, it reaches its maximum at the point $(x_1, t_1) \in D$ (this follows from the assumption that $T(x, t)$ attains maximum on $\bar{D}$). We will show that U cannot have the maximum in D.

This will imply that the hypothesis $(x^*, t^*) \in D$ is wrong and the assertion will be proved. As in (2.4) and (2.5), $\frac{\partial^2 U}{\partial x^2}(x_1, t_1) \leqslant 0$, $\frac{\partial^\alpha U}{\partial t^\alpha}(x_1, t_1) \geqslant 0$, $\frac{\partial^\beta U}{\partial t^\beta}(x_1, t_1) \geqslant 0$. We have

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2}(x_1, t_1) &= \frac{\partial^2 T}{\partial x^2}(x_1, t_1), \\
\frac{\partial^\alpha U}{\partial t^\alpha}(x_1, t_1) &= \frac{\partial^\alpha T}{\partial t^\alpha}(x_1, t_1) - k\Gamma(1+\alpha), \\
\frac{\partial^\beta U}{\partial t^\beta}(x_1, t_1) &= \frac{\partial^\beta T}{\partial t^\beta}(x_1, t_1) - k\frac{\Gamma(1+\alpha)}{\Gamma(\alpha+1-\beta)} t_1^{\alpha-\beta}.
\end{aligned} \tag{2.6}$$

Thus,

$$\tau\frac{\partial^\alpha U}{\partial t^\alpha}(x_1, t_1) + \frac{\partial^\beta U}{\partial t^\beta}(x_1, t_1) - \mathcal{D}\frac{\partial^2 U}{\partial x^2} \geqslant 0$$

since at (x_1, t_1) we have the maximum.

From (2.6) and (2.1),

$$\tau\frac{\partial^\alpha T}{\partial t^\alpha}(x_1, t_1) + \frac{\partial^\beta T}{\partial t^\beta}(x_1, t_1) - \mathcal{D}\frac{\partial^2 T}{\partial x^2} = k\Gamma(1+\alpha)\left[1 + \frac{t_1^{\alpha-\beta}}{\Gamma(\alpha+1-\beta)}\right] > 0.$$

This is a contradiction, $(x_1, t_1) \notin D$ and thus $(x^*, t^*) \in \gamma$. □

Remark 2. The condition that $T(x, t)$ is continuous in $\overline{D}$ can be relaxed by requiring that $T(x, t)$ is continuous in D (what we know since $T(x, t)$ satisfies (2.1)) and that for $(x, t) \in \gamma$ the limit inferior $\underline{\lim} T(x, t)$ exists (see [14]). Thus, we have

Proposition 3. *If for $T(x, t)$, $\underline{\lim} T(x, t)$ exists for every $(x, t) \in \gamma$, then $T(x, t)$ reaches its maximum and minimum on γ.*

Proof. We will assume that $\lim_{(x,t)\to(x_0,t_0)} T(x, t) \geqslant 0$, for all $(x_0, t_0) \in \gamma$ and by this, we will show that $T(x, t) \geqslant 0$, $(x, t) \in D$. This implies the assertion in the case of the non-negative minimum and thus we will have the assertion in the arbitrary case.

Let us assume that $\lim_{(x,t)\to(x_0,t_0)} T(x, t) \geqslant 0$, for all $(x_0, t_0) \in \gamma$. By the definition of the limit inferior, this means that for all $\varepsilon > 0$, there exists δ_ε such that $(x, t) \in L_{\delta_\varepsilon}((x_0, t_0), \delta_\varepsilon)$ implies $T(x, t) \geqslant -\varepsilon$, where $L((x_0, t_0), \delta_\varepsilon)$ denotes a ball with the radius δ_ε centred at (x_0, t_0). Fix ε and for every $(x_0, t_0) \in \gamma$, find L_{δ_ε}. By the compactness, we have that there exist finitely many balls with radii δ_{ε_i}, $i = 1, \ldots, n$, which cover γ; find $\delta = \min_{i=1,\ldots,n} \delta_{\varepsilon_i}$. Consider a rectangle $D_0 \subset D$ with the boundary γ_0 that lies at the distance $\delta_0 < \delta$ from γ so that $T(x, t) \geqslant -\varepsilon$, $(x, t) \in D_0$. Letting $\varepsilon \to 0$, we obtain $T(x, t) \geqslant 0$ in D. □

Theorem 1 could be used to prove uniqueness problem (2.1) and (2.2). We state this as

Proposition 4. *The solution to the initial boundary value problem (2.1) and (2.2) for $0 \leqslant \beta \leqslant \alpha \leqslant 1$ is unique.*

Proof. Let $T_1(x, t)$ and $T_2(x, t)$ be two solutions to (2.1) and (2.2). Applying theorem 1 to $T(x, t) = T_1(x, t) - T_2(x, t)$, we obtain that $T_1(x, t) = T_2(x, t)$. □

2.2. The case of bounded domain

We present next a solution to (2.1) and (2.2) in the special case when $f_1(t) = f_2(t) = 0$. For the case $\tau = 0$, the problem was treated in [12, 15] by different methods. Assuming that a solution has a form $T(x,t) = X(x)U(t)(t > 0, x \in (0,l))$ and using the standard procedure, separation of variables, we obtain

$$T(x,t) = \sum_{k=1}^{\infty} U_k(0) F_k(t) \sin \lambda_k x, \qquad U_k(0) = \frac{2}{\ell} \int_0^{\ell} g(x) \sin \lambda_k x \, \mathrm{d}x, \tag{2.7}$$

where $\lambda_k = \frac{k\pi}{l}$, and $F_k(t)$ satisfies

$$\tau \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} F_k(t) + \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} F_k(t) + \lambda_k^2 \mathcal{D} F_k(t) = 0, \tag{2.8}$$

subject to

$$F_k(0) = 1, \qquad \left[\frac{\mathrm{d}}{\mathrm{d}t} F_k(t)\right]_{t=0} = 0. \tag{2.9}$$

The Laplace transform $\mathcal{L}\{F_k(t)\}(s) = \overline{F_k}(s) = \int_0^\infty \exp(-st) F_k(t)\,\mathrm{d}t$ of (2.7) (with $\mathrm{Re}\, s > s_0$ which will be discussed later) leads to

$$\overline{F_k}(s) = \frac{\tau s^{\alpha-1} + s^{\beta-1}}{\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}} = \frac{1}{s} - \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)}. \tag{2.10}$$

Expanding the right-hand side of (2.10) in a power series of s (see [[16], p 155]) we obtain

$$\overline{F_k}(s) = [\tau s^{\alpha-1} + s^{\beta-1}] \frac{1}{\lambda_k^2 \mathcal{D}} \frac{\lambda_k^2 \mathcal{D} s^{-\beta}}{\tau s^{\alpha-\beta} + 1} \frac{1}{1 + \frac{\lambda_k^2 \mathcal{D} s^{-\beta}}{\tau s^{\alpha-\beta}+1}}. \tag{2.11}$$

Since

$$\frac{1}{1 + \frac{\lambda_k^2 \mathcal{D} s^{-\beta}}{\tau s^{\alpha-\beta}+1}} = \sum_{j=0}^{\infty} (-1)^j \left(\frac{\lambda_k^2 \mathcal{D} s^{-\beta}}{\tau s^{\alpha-\beta} + 1}\right)^j,$$

we have

$$\overline{F_k}(s) = \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda_k^2 \mathcal{D})^j [\tau s^{(\alpha-\beta)-(1+j\beta)} + s^{-(1+j\beta)}]}{(\tau s^{\alpha-\beta} + 1)^{(1+j)}}. \tag{2.12}$$

The series in (2.12) is convergent if we choose $s_0 > 0$ so that

$$\left|\frac{\lambda_k^2 \mathcal{D} s^{-\beta}}{1 + \tau s^{\alpha-\beta}}\right| = \left|\frac{\lambda_k^2 \mathcal{D}}{s^\beta + \tau s^\alpha}\right| < 1$$

holds if $\mathrm{Re}\, s > s_0$. This determines $s_0 > 0$. From [[16], p 21], we have

$$\mathcal{L}^{-1}\left\{\sum_{j=0}^{\infty} \frac{1}{j!} \frac{(-1)^j \left(\lambda_k^2 \mathcal{D}\right)^j}{(\tau)^j} \frac{j![s^{(\alpha-\beta)-(1+j\beta)}]}{\left(s^{\alpha-\beta} + \frac{1}{\tau}\right)^{(1+j)}}\right\}$$
$$= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{(-1)^j \left(\lambda_k^2 \mathcal{D}\right)^j}{(\tau)^j} t^{\alpha j} E^{(j)}_{\alpha-\beta,(1+j\beta)}\left(-\frac{t^{(\alpha-\beta)}}{\tau}\right);$$

$$\mathcal{L}^{-1}\left\{\sum_{j=0}^{\infty} \frac{1}{j!} \frac{(-1)^j \left(\lambda_k^2 \mathcal{D}\right)^j}{(\tau)^{j+1}} \frac{j![s^{-(1+j\beta)}]}{\left(s^{\alpha-\beta} + \frac{1}{\tau}\right)^{(1+j)}}\right\}$$
$$= \sum_{j=0}^{\infty} \frac{1}{j!} \frac{(-1)^j \left(\lambda_k^2 \mathcal{D}\right)^j}{(\tau)^{j+1}} t^{(\alpha+1)j-\beta} E^{(j)}_{\alpha-\beta,(\alpha+1+\beta(j-1))}\left(-\frac{t^{(\alpha-\beta)}}{\tau}\right).$$

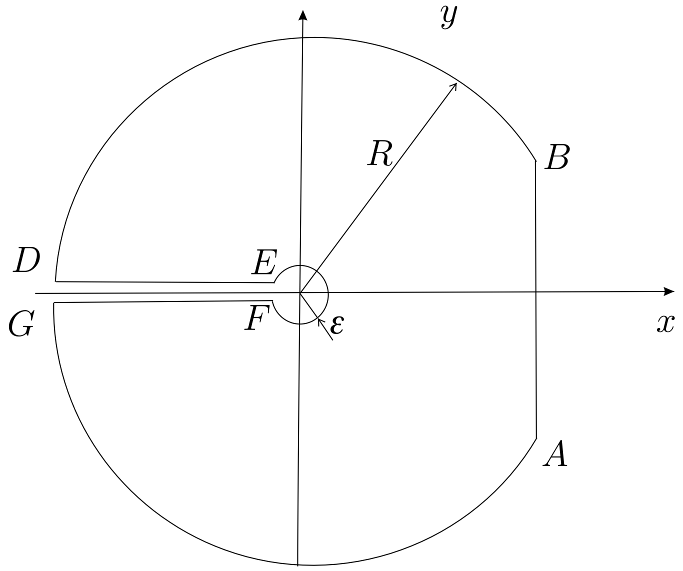

Figure 1. Integration contour for the inverse Laplace transform of (2.16).

So, the solution $F_k(t)$, $t > 0$, has the form

$$F_k(t) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{(-1)^j \left(\lambda_k^2 \mathcal{D}\right)^j}{(\tau)^j} t^{\alpha j} E^{(j)}_{\alpha-\beta,(1+j\beta)} \left(-\frac{t^{(\alpha-\beta)}}{\tau}\right)$$
$$+ \sum_{j=0}^{\infty} \frac{1}{j!} \frac{(-1)^j \left(\lambda_k^2 \mathcal{D}\right)^j}{(\tau)^{j+1}} t^{(\alpha+1)j-\beta} E^{(j)}_{\alpha-\beta,(\alpha+1+\beta(j-1))} \left(-\frac{t^{(\alpha-\beta)}}{\tau}\right). \tag{2.13}$$

For numerical purposes, we can write the inversion of (2.10) as follows. First, we expand (2.10) into a power series as

$$\overline{F}_k(s) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} (-1)^n \binom{n}{p} \left(\lambda_k^2 D\right)^p \tau^{n-p} (\tau s^{(\alpha-\beta)(n+1)-\alpha p-1} + s^{(\alpha-\beta)n-\alpha p-1}). \tag{2.14}$$

Applying a term by term inversion, we obtain ($t > 0$)

$$F_k(t) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} (-1)^n \binom{n}{p} \left(\lambda_k^2 \mathcal{D}\right)^p \tau^{n-p} \left[\frac{\tau t^{-(\alpha-\beta)(n+1)+\alpha p}}{\Gamma[-(\alpha-\beta)(n+1)+\alpha p+1]}\right.$$
$$\left. + \frac{t^{-(\alpha-\beta)n+\alpha p}}{\Gamma[-(\alpha-\beta)n+\alpha p+1]}\right]. \tag{2.15}$$

Next, we express the inversion to (2.10) in terms of an appropriate contour integral. We have

$$F_k(t) = H(t) - \frac{1}{2\pi\,\mathrm{i}} \int_\gamma \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \mathrm{e}^{st}\, \mathrm{d}s, \qquad t > 0, \tag{2.16}$$

where $H(t)$ is the Heaviside step function (equals 1 for $t > 0$) and γ is given as $\gamma = \{s; \mathrm{Re}\, s = \sigma, \sigma > 0\}$. Let γ_0 be the closed contour shown in figure 1.

Cauchy formula gives

$$\int_{\gamma_0} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)}\, \mathrm{d}s = 2\pi\,\mathrm{i} \sum \mathrm{Re}\, s \left\{\frac{\mathrm{e}^{st} \lambda_k^2 \mathcal{D}}{s(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D})}\right\}. \tag{2.17}$$

Integrals

$$\int_{BD} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \, \mathrm{d}s \qquad \text{and} \qquad \int_{GA} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \, \mathrm{d}s$$

tend to zero as $R \to \infty$, so they do not contribute to the left-hand side of (2.17). What contributes are integrals

$$\int_{AB} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D})} \, \mathrm{d}s, \qquad \int_{DE} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D})} \, \mathrm{d}s,$$

$$\int_{EF} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D})} \, \mathrm{d}s, \qquad \int_{FG} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D})} \, \mathrm{d}s.$$

Note that $\frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)}$ is analytic within γ_0. To prove this, suppose that there is a solution to the equation

$$\tau s^\alpha + s^\beta + \lambda_k^2 D = 0. \tag{2.18}$$

Let $z = r \exp(\mathrm{i}\phi)$ be a solution to (2.18). Separating real and imaginary parts in (2.18), we obtain

$$\begin{aligned} &\tau r^\alpha \cos(\alpha\phi) + r^\beta \cos(\beta\phi) + \lambda_k^2 D = 0 \\ &\tau r^\alpha \sin(\alpha\phi) + r^\beta \sin(\beta\phi) = 0. \end{aligned} \tag{2.19}$$

From $(2.19)_2$, it follows that solutions are complex conjugate, i.e., if z is a solution so is $\bar{z}$. Also, since $0 \leqslant \beta \leqslant \alpha \leqslant 1$, we have that $\sin(\alpha\phi)$ and $\sin(\beta\phi)$ are of the same sign, so that (2.19) cannot be satisfied. Thus, there are no solutions to (2.18). Therefore, within γ_0, we have $\sum \operatorname{Re} s\left\{\mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D})}\right\} = 0$. Then

$$\int_{\gamma_0} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \, \mathrm{d}s = 0,$$

or

$$\begin{aligned} \lim_{R\to\infty,\varepsilon\to 0} \Bigg(\int_{DE} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \, \mathrm{d}s + \int_{EF} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \, \mathrm{d}s \\ + \int_{FG} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \, \mathrm{d}s \Bigg) = 2\pi \, \mathrm{i}[F_k(t) - H(t)]. \end{aligned} \tag{2.20}$$

To determine $F_k(t)$, we let $s = r\,\mathrm{e}^{\mathrm{i}\pi}$ on DE and $s = r\,\mathrm{e}^{-\mathrm{i}\pi}$ on FG, and $s = \varepsilon\,\mathrm{e}^{\theta\,\mathrm{i}}, \theta \in (\pi, -\pi), \varepsilon > 0$ on EF. Then, from (2.20), we have

$$\begin{aligned} &\int_{DE} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \, \mathrm{d}s = -\lambda_k^2 \mathcal{D} \int_\varepsilon^R \mathrm{e}^{-rt} \\ &\qquad \times \frac{\mathrm{d}r}{r[\tau r^\alpha \cos\alpha\pi + r^\beta \cos\beta\pi + \lambda_k^2 \mathcal{D} + \mathrm{i}(\tau r^\alpha \sin\alpha\pi + r^\beta \sin\beta\pi)]}; \\ &\int_{FG} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D})} \, \mathrm{d}s = \lambda_k^2 \mathcal{D} \int_\varepsilon^R \mathrm{e}^{-rt} \\ &\qquad \times \frac{\mathrm{d}r}{r[\tau r^\alpha \cos\alpha\pi + r^\beta \cos\beta\pi + \lambda_k^2 \mathcal{D} - \mathrm{i}(\tau r^\alpha \sin\alpha\pi + r^\beta \sin\beta\pi)]}; \\ &\int_{EF} \mathrm{e}^{st} \frac{\lambda_k^2 \mathcal{D}}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \, \mathrm{d}s = -\mathrm{i}\lambda_k^2 \mathcal{D} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\varepsilon t(\cos\theta + \mathrm{i}\sin\theta)} \, \mathrm{d}\theta}{\tau\varepsilon^\alpha \,\mathrm{e}^{\mathrm{i}\alpha\theta} + \varepsilon^\beta \,\mathrm{e}^{\mathrm{i}\beta\theta} + \lambda_k^2 \mathcal{D}}. \end{aligned} \tag{2.21}$$

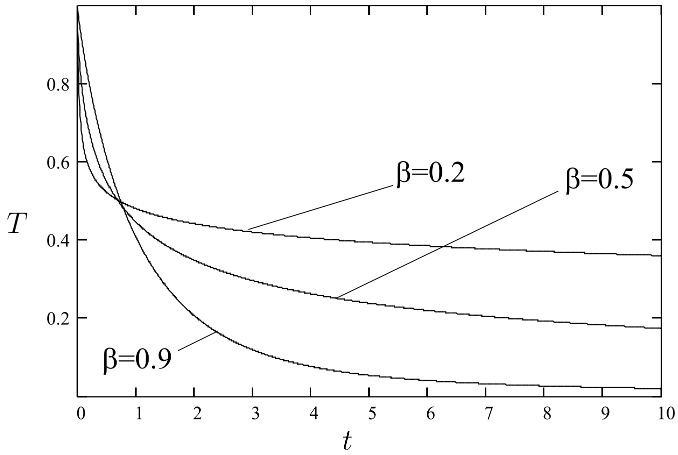

Figure 2. Solution of (2.24) for $\alpha = 0.9$ and $\beta = 0.2, 0.5, 0.9$.

Combining terms in (2.21) and taking limit when $R \to \infty$, $\varepsilon \to 0$, we obtain

$$F_k(t) = \frac{\lambda_k^2 \mathcal{D}}{\pi} \int_0^\infty \frac{(\tau r^\alpha \sin \alpha\pi + r^\beta \sin \beta\pi)\, \mathrm{e}^{-rt}\, \mathrm{d}r}{rK}, \tag{2.22}$$

where

$$K = \tau^2 r^{2\alpha} + r^{2\beta} + \left(\lambda_k^2 \mathcal{D}\right)^2 + 2\lambda_k^2 \mathcal{D}(\tau r^\alpha \cos \alpha\pi + r^\beta \cos \beta\pi) + 2\tau r^{\alpha+\beta} \cos \pi(\alpha - \beta). \tag{2.23}$$

With $F_k(t)$ determined by (2.15) or (2.22), the complete solution of the problem becomes

$$\begin{aligned} T(x,t) &= \sum_{k=0}^{\infty} U_k(0) \left\{ \sum_{n=0}^{\infty} \sum_{p=0}^{n} (-1)^n \binom{n}{p} \left(\lambda_k^2 \mathcal{D}\right)^p \tau^{n-p} \right. \\ &\quad \left. \times \left[\frac{\tau t^{-(\alpha-\beta)(n+1)+\alpha p}}{\Gamma[-(\alpha-\beta)(n+1)+\alpha p+1]} + \frac{t^{-(\alpha-\beta)n+\alpha p}}{\Gamma[-(\alpha-\beta)n+\alpha p+1]} \right] \right\} \sin(\lambda_k x) \\ &= \sum_{k=0}^{\infty} U_k(0) \left\{ \frac{\lambda_k^2 \mathcal{D}}{\pi} \int_0^\infty \frac{(\tau r^\alpha \sin \alpha\pi + r^\beta \sin \beta\pi)\, \mathrm{e}^{-rt}\, \mathrm{d}r}{rK} \right\} \sin(\lambda_k x). \end{aligned} \tag{2.24}$$

2.3. Numerical results

We present numerical results for (2.24) in the special case $\alpha = 0.9$ and for three values of β, $\beta = 0.2, 0.5, 0.9$. We used the integral representation in (2.24) and the following special values of parameters: $\tau = 0.1$, $U_k(0) = 2$ for $k = 1$ all other $U_k(0) = 0$. Also, we use $\lambda_1 = 1$, $D = 1$ and $x = \pi/2$. The results are shown in figure 2.

3. Properties and solutions of problems for $0 < \beta \leqslant \alpha \leqslant 2$

We shall treat signalling and Cauchy problem for the generalized telegraph equation (1.5) in the cases when $\alpha > 1$.

3.1. Signalling problem

Consider (1.5)

$$\tau\frac{\partial^\alpha T}{\partial t^\alpha}+\frac{\partial^\beta T}{\partial t^\beta}=\mathcal{D}\frac{\partial^2 T}{\partial x^2},\qquad x\in(0,\infty),\qquad t>0,\tag{3.1}$$

subject to

$$\begin{aligned}&T(x,0^+)=0, &&\frac{\partial T(x,0^+)}{\partial t}=0, && x\in(0,\infty),\\ &\lim_{x\to 0}T(x,t)=\delta(t), &&\lim_{x\to\infty}T(x,t)=0, && t>0.\end{aligned}\tag{3.2}$$

Applying the Laplace transform with respect to time t, we obtain the following differential equation for the Laplace transform of $T(x,t)$, with $\mathrm{Re}\, s>0$,

$$\frac{\partial^2}{\partial x^2}\overline{T}(x,s)-\frac{\tau s^\alpha+s^\beta}{\mathcal{D}}\overline{T}(x,s)(x,s)=0,\tag{3.3}$$

subject to

$$\lim_{x\to\infty}\overline{T}(x,s)=0,\qquad \lim_{x\to 0}\overline{T}(x,s)=1.\tag{3.4}$$

From (3.3) and (3.4), we obtain ($\mathrm{Re}\, s>0$)

$$\overline{T}(x,s)=\mathrm{e}^{-\frac{x}{\sqrt{\mathcal{D}}}\sqrt{\tau s^\alpha+s^\beta}}.\tag{3.5}$$

We present the inversion of (3.5) in the form of a series and in the integral form. From (3.5), we have

$$\begin{aligned}\overline{T}(x,s)&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x}{\sqrt{\mathcal{D}}}\right)^n s^{\frac{\beta n}{2}}(1+\tau s^{\alpha-\beta})^{\frac{n}{2}}\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{x}{\sqrt{\mathcal{D}}}\right)^n\frac{\tau^k}{n!k!}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1-k\right)}s^{(\alpha-\beta)k+\frac{\beta n}{2}},\end{aligned}\tag{3.6}$$

where we used the following series representation: $(1+x)^\gamma=\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+1)}{k!\Gamma(\gamma+1-k)}x^k$, $\gamma>0$. Term by term inversion of (3.6) leads to

$$T(x,t)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{x}{\sqrt{\mathcal{D}}}\right)^n\frac{\tau^k}{n!k!}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1-k\right)}\frac{t^{-[(\alpha-\beta)k+\frac{\beta n}{2}+1]}}{\Gamma\left(-\left[(\alpha-\beta)k+\frac{\beta n}{2}\right]\right)}.\tag{3.7}$$

To express the solution in the integral form, we additionally assume $0<\alpha-\beta\leqslant 1$. Since $T(x,t)=\mathcal{L}^{-1}\{\overline{T}(x,s)\}=\frac{1}{2\pi\,\mathrm{i}}\int_\gamma \mathrm{e}^{-\frac{x}{\sqrt{\mathcal{D}}}\sqrt{\tau s^\alpha+s^\beta}}\,\mathrm{e}^{st}\,\mathrm{d}s$, where γ is the same contour as the one shown in figure 1. In the case $0<\alpha-\beta\leqslant 1$, the only branch point of $\sqrt{\tau s^\alpha+s^\beta}$ is $s=0$. Therefore, the function $\mathrm{e}^{-\frac{x}{\sqrt{\mathcal{D}}}\sqrt{\tau s^\alpha+s^\beta}}\,\mathrm{e}^{st}$ is analytic within γ_0. Applying the same procedure as in the previous case, we obtain

$$T(x,t)=\frac{1}{\pi}\int_0^\infty\sin\left[\frac{x}{\sqrt{\mathcal{D}}}\sqrt{R}\sin\frac{\phi}{2}\right]\mathrm{e}^{-\frac{x}{\sqrt{\mathcal{D}}}\sqrt{R}\cos\frac{\phi}{2}-pt}\,\mathrm{d}p,\tag{3.8}$$

where

$$\begin{aligned}R&=\sqrt{\tau^2p^{2\alpha}+p^{2\beta}+2\tau p^{\alpha+\beta}\cos[\pi(\alpha-\beta)]},\\ \phi&=\mathrm{arctg}\frac{\tau p^{\alpha-\beta}\sin\pi\alpha+\sin\pi\beta}{\tau p^{\alpha-\beta}\cos\pi\alpha+\cos\pi\beta}.\end{aligned}\tag{3.9}$$

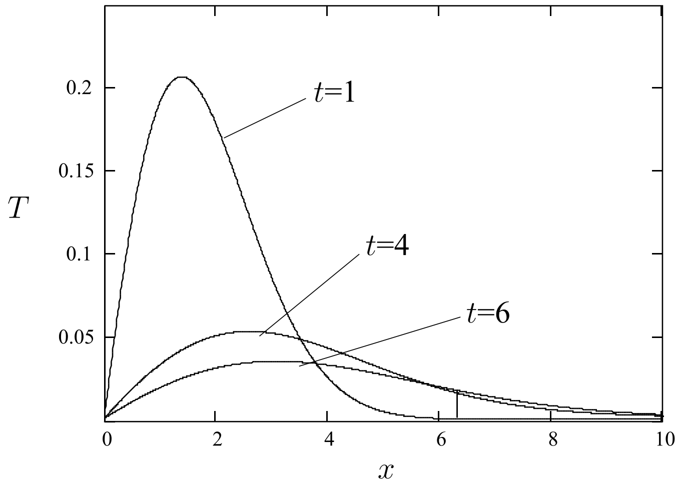

Figure 3. Solution given by (3.8) for $\alpha = 1.8$, $\beta = 0.9$, $\tau = 0.01$, $\mathcal{D} = 1$.

Note that (3.7) is valid for any α, β while (3.8) holds for $0 < \alpha - \beta \leqslant 1$. We shall compare numerical results obtained according to (3.7) and (3.8) with the concrete values of α, β, when these formulae are applicable.

3.1.1. Numerical results. First, we consider the case $\alpha = 1.8$, $\beta = 0.9$, $\tau = 0.01$, $\mathcal{D} = 1$. The function $T(x, t)$ calculated by using (3.7) and (3.8) is shown in figure 3 for three different time instants.

3.2. Cauchy problem

Finally, we consider (1.5) and (1.7) with $\frac{\partial T}{\partial t}(x, 0^+) = 0$. For arbitrary $g(x)$, the solution is given by (1.8). Therefore, we determine the Green function $\mathcal{G}_c(x, t)$. Applying the Laplace transform to (1.5), we obtain ($\mathrm{Re}\, s > 0$)

$$\frac{\partial^2}{\partial x^2}\overline{\mathcal{G}_c}(x, s) - \frac{\tau s^\alpha + s^\beta}{\mathcal{D}}\overline{\mathcal{G}_c}(x, s) = -\frac{\tau s^{\alpha-1} + s^{\beta-1}}{\mathcal{D}}\delta(x). \tag{3.10}$$

The differential equation (3.10) can be easily solved by using the same procedure as in [5]. Thus, we obtain

$$\overline{\mathcal{G}_c}(x, s) = \frac{1}{2\sqrt{\mathcal{D}}}\frac{\sqrt{\tau s^\alpha + s^\beta}}{s}\,\mathrm{e}^{\frac{-|x|}{\sqrt{\mathcal{D}}}\sqrt{\tau s^\alpha + s^\beta}}. \tag{3.11}$$

The inverse Laplace transform of (3.11) is obtained in the form of a power series. Thus, we expand (3.11) to obtain

$$\overline{\mathcal{G}_c}(x, s) = \frac{1}{2\sqrt{\mathcal{D}}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{|x|}{\sqrt{\mathcal{D}}}\right)^n \frac{\tau^k}{n!k!}\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+3}{2} - k\right)} s^{(\alpha-\beta)k + \beta\frac{n+1}{2} - 1}. \tag{3.12}$$

Term by term inversion leads to

$$\mathcal{G}_c(x, t) = \frac{1}{2\sqrt{\mathcal{D}}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{|x|}{\sqrt{\mathcal{D}}}\right)^n \tau^k \frac{\Gamma\left(\frac{n+3}{2}\right)}{n!k!\Gamma\left(\frac{n+3}{2} - k\right)}\frac{t^{-\left((\alpha-\beta)k + \beta\frac{n+1}{2}\right)}}{\Gamma\left(-\left[(\alpha-\beta)k + \beta\frac{n+1}{2} - 1\right]\right)}. \tag{3.13}$$

To check the numerical accuracy of (3.13), we derive the connection between the solution of the Cauchy and signalling problems. The Laplace transforms which correspond to these two problems are given by (3.5) and (3.11). Differentiating (3.5) with respect to s, $\operatorname{Re} s > 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\overline{T}(x,s) = -\frac{x}{2\sqrt{\mathcal{D}}}\frac{\alpha\tau s^{\alpha-1}+\beta s^{\beta-1}}{\sqrt{\tau s^{\alpha}+s^{\beta}}}\,\mathrm{e}^{-\frac{x}{\sqrt{\mathcal{D}}}\sqrt{\tau s^{\alpha}+s^{\beta}}}. \tag{3.14}$$

By comparing (3.11) and (3.14), we conclude that

$$x\overline{\mathcal{G}_c}(x,s) = \frac{\alpha-\beta}{\alpha\beta}s\frac{s^{\alpha-\beta-1}}{s^{\alpha-\beta}+\frac{\beta}{\alpha\tau}}\frac{\mathrm{d}}{\mathrm{d}s}\overline{T}(x,s) - \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}s}\overline{T}(x,s). \tag{3.15}$$

We will use the following properties of the Laplace transform in (3.15): $-\frac{\mathrm{d}}{\mathrm{d}s}\overline{T}(x,s) = \mathcal{L}\{tT(x,t)\}$, $\frac{s^{\sigma-1}}{s^{\sigma}+\lambda} = \mathcal{L}\{E_{\sigma}(-\lambda t^{\sigma})\}$, $s\tilde{f}(s) = \mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\} + f(0)$, where $E_{\sigma}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\sigma n+1)}$ is the Mittag–Leffler function. We obtain

$$x\mathcal{G}_c(x,t) = \frac{t}{\beta}T(x,t) - \frac{\alpha-\beta}{\alpha\beta}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\sigma}[\sigma T(x,\sigma)]E_{\alpha-\beta}\left(-\frac{\beta}{\alpha\tau}(t-\sigma)^{\alpha-\beta}\right)\mathrm{d}\sigma. \tag{3.16}$$

Different grouping of terms in (3.15) leads to another relation between $T(x,t)$ and $\mathcal{G}_c(x,t)$ given as

$$x\mathcal{G}_c(x,t) = \frac{t}{\alpha}T(x,t) - \frac{\alpha-\beta}{\alpha\beta}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[E_{\alpha-\beta}\left(-\frac{\beta}{\alpha\tau}\sigma^{\alpha-\beta}\right)\right](t-\sigma)T(x,t-\sigma)\,\mathrm{d}\sigma. \tag{3.17}$$

Finally, by using the solution to the signalling problem given in the form (3.8) in (3.17), we have

$$\begin{aligned}\mathcal{G}_c(x,t) &= \frac{1}{\alpha}\frac{t}{\pi x}\int_0^{\infty}\sin\left[\frac{x}{\sqrt{\mathcal{D}}}\sqrt{R}\sin\frac{\phi}{2}\right]\mathrm{e}^{-\frac{x}{\sqrt{\mathcal{D}}}\sqrt{R}\cos\frac{\phi}{2}-pt}\,\mathrm{d}p\\ &\quad - \frac{\alpha-\beta}{\alpha\beta}\frac{1}{\pi x}\int_0^t\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[E_{\alpha-\beta}\left(-\frac{\beta}{\alpha\tau}\sigma^{\alpha-\beta}\right)\right](t-\sigma)\\ &\quad \times\left[\int_0^{\infty}\sin\left[\frac{x}{\sqrt{\mathcal{D}}}\sqrt{R}\sin\frac{\phi}{2}\right]\mathrm{e}^{-\frac{x}{\sqrt{\mathcal{D}}}\sqrt{R}\cos\frac{\phi}{2}-p(t-\sigma)}\,\mathrm{d}p\right]\mathrm{d}\sigma,\end{aligned} \tag{3.18}$$

where R and ϕ are given by (3.9).

3.2.1. Numerical results. We present in figure 4 the solution $\mathcal{G}_c(x,t)$ obtained by using (3.13) for special values of parameters: $\mathcal{D}=1$, $\tau = 0.001$, $\alpha = 2$, $\beta = 1.25$, $t = 2$.

We show in figure 5 the solution to (1.5) and (1.7) with $g(x) = \delta(x)$ and $\mathcal{D}=1$, $\tau = 0.001$, $\alpha = 2$, $\beta = 1.5$, $t = 2$.

Finally, we show in figure 6 the solution of (1.5) and (1.7) with $g(x) = \delta(x)$ and $\mathcal{D}=1$, $\tau = 0.001$, $\alpha = 2$, $\beta = 1.9$, $t = 2$.

3.3. The case of bounded domain

We present a solution to (2.1) and (2.2) in the special case when $f_1(t) = f_2(t) = 0$ and $0 \leqslant \beta \leqslant \alpha \leqslant 2$. The only difference consists in inverting (2.10) with different values of α and β. First, we write (2.10) as

$$\overline{F_k}(s) = \frac{1}{s} - \frac{\lambda_k^2\mathcal{D}}{s\left(\tau s^{\alpha}+s^{\beta}+\lambda_k^2\mathcal{D}\right)}. \tag{3.19}$$

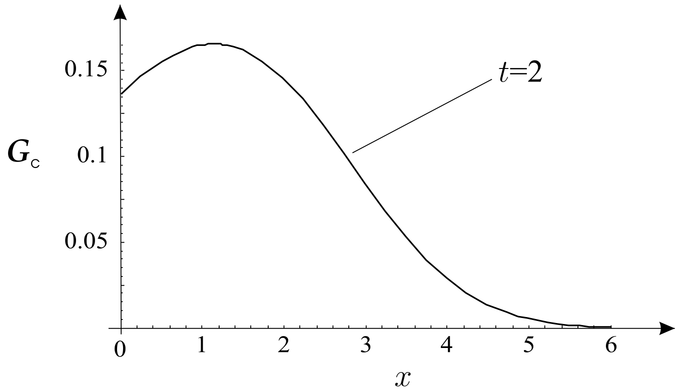

Figure 4. Solution given by (3.13) for $\alpha = 2$, $\beta = 1.25$, $\tau = 0.001$, $\mathcal{D} = 1$.

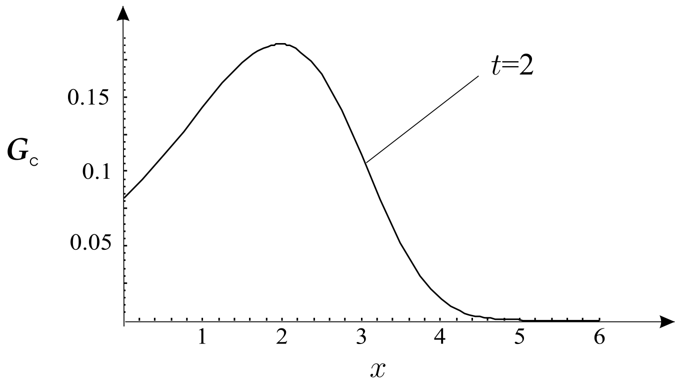

Figure 5. Solution to (1.5) and (1.7) for $\alpha = 2$, $\beta = 1.5$, $\tau = 0.001$, $\mathcal{D} = 1$, $t = 2$, $g(x) = \delta(x)$.

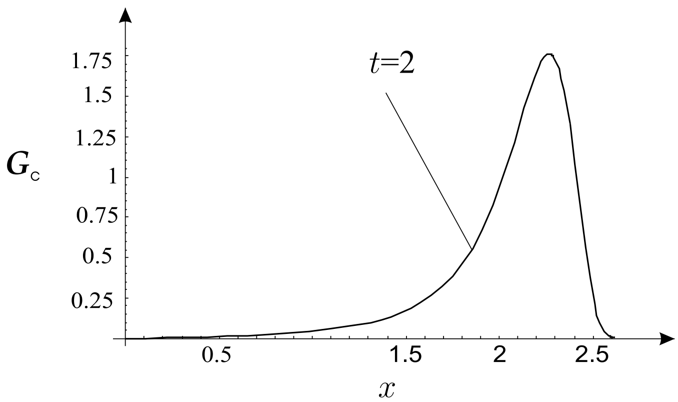

Figure 6. Solution to (1.5) and (1.7) for $\alpha = 2$, $\beta = 1.9$, $\tau = 0.001$, $\mathcal{D} = 1$, $t = 2$, $g(x) = \delta(x)$.

The function $F_k(t)$ is given as

$$F_k(t) = H(t) - \frac{\lambda_k^2 \mathcal{D}}{2\pi \,\mathrm{i}} \int_\gamma \frac{1}{s\left(\tau s^\alpha + s^\beta + \lambda_k^2 \mathcal{D}\right)} \mathrm{e}^{st} \, \mathrm{d}s, \tag{3.20}$$

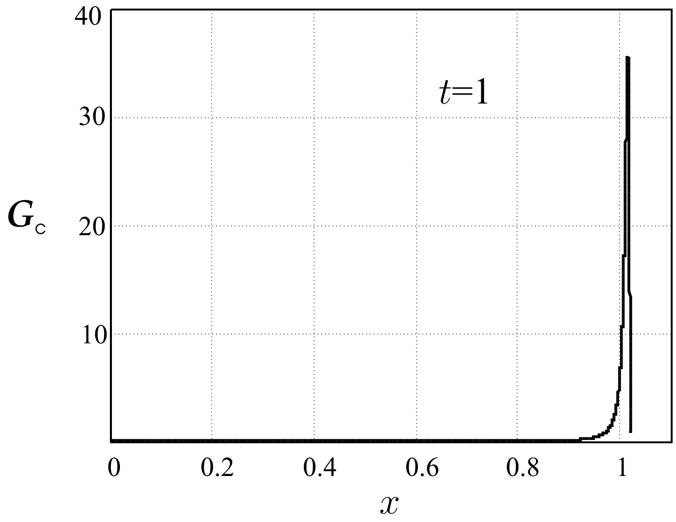

Figure 7. Solution to (1.5) and (1.7) for $\mathcal{D} = (1+\tau)$, $\alpha = \beta = 1.995$ and $t = 1$.

where γ is shown in figure 1. We shall evaluate (3.20) by using the contour integral along γ_0. Note that the function under integral sign is not analytic within γ_0, since the function $\tau s^\alpha + s^\beta + \lambda_k^2\mathcal{D}$ has zeros within γ_0. By the argument principle, the function $\tau s^\alpha + s^\beta + \lambda_k^2\mathcal{D}$ has exactly two (conjugate) zeros. Since, for $\mathrm{Re}\, s > 0$, the imaginary part $\mathrm{Im}(\tau s^\alpha + s^\beta + \lambda_k^2\mathcal{D}) = \tau R^\alpha \sin\alpha\theta + R^\beta \sin\beta\theta > 0$, it follows that both zeros lie in the left part of the complex plane $\mathrm{Im}\, s < 0$. Let s_0 and $\bar{s}_0$ be zeros of $\tau s^\alpha + s^\beta + \lambda_k^2\mathcal{D}$. Then, Cauchy's formula gives

$$\int_{\gamma_0} \mathrm{e}^{st}\overline{F}_k(s)\,\mathrm{d}s = 2\pi\,\mathrm{i}(\mathrm{Re}\, s\{\mathrm{e}^{s_0 t}\overline{F}_k(s_0)\} + \mathrm{Re}\, s\{\mathrm{e}^{\bar{s}_0 t}\overline{F}_k(\bar{s}_0)\}). \tag{3.21}$$

The contour integral in (3.21) can be evaluated in the similar way as in the case when there are no singularity, so that $F_k(t)$ reads

$$F_k(t) = \frac{\lambda_k^2\mathcal{D}}{2\pi\,\mathrm{i}}\int_0^\infty \left[\frac{1}{\tau p^\alpha\,\mathrm{e}^{-\mathrm{i}\pi\alpha} + p^\beta\,\mathrm{e}^{-\mathrm{i}\pi\beta} + \lambda_k^2\mathcal{D}} - \frac{1}{\tau p^\alpha\,\mathrm{e}^{\mathrm{i}\pi\alpha} + p^\beta\,\mathrm{e}^{\mathrm{i}\pi\beta} + \lambda_k^2\mathcal{D}}\right]\frac{\mathrm{e}^{-pt}}{p}\,\mathrm{d}p$$
$$- \lambda_k^2\mathcal{D}(\mathrm{Re}\, s\{\mathrm{e}^{s_0 t}\overline{F}_k(s_0)\} + \mathrm{Re}\, s\{\mathrm{e}^{\bar{s}_0 t}\overline{F}_k(\bar{s}_0)\}). \tag{3.22}$$

4. Conclusion

In this work, we studied the fractional telegraph equation in finite and infinite spatial domain. We treated signalling and Cauchy problems. Our main results may be summarized as follows.

(1) For the case when $0 < \alpha, \beta \leqslant 1$ in (1.5), we proved an analogue of the classical maximum principle for the heat equation. This result leads to the proof of uniqueness for the initial-boundary value problem (2.1) and (2.2).

(2) For the case $0 < \alpha, \beta \leqslant 1$ and $f_1(t) = f_2(t) = 0$ in (2.1) and (2.2), we obtained the solution in the form (2.24). As a matter of fact, in (2.24), we presented the solution in two forms. In one form the time evolution is presented in terms of a power series, while

in the second form of solution the time evolution is presented in an integral form. The two forms of solution helped us in numerical examples. Namely, when it was possible (that is when both expressions were convergent), we compared the results and found good agreements. In other cases, we used the form that leads to a bounded solution.

(3) For the case $0 < \beta \leqslant \alpha \leqslant 2$, equation (1.5) was solved for signalling and Cauchy problems. The solutions are obtained in the forms (3.7) and (3.8) for the signalling problem. Again, in numerical applications, both forms of the solutions are used. For the Cauchy problem, the solution is given in the form (3.13) and (3.18). It is known that in the case of a single fractional derivative, there is a relation between solutions of Cauchy and signalling problems (see [[8], p 327]). We derived a generalization of this relation given by (3.16) and (3.17). Note that for the case when $\alpha = \beta$, both relations lead to

$$x\mathcal{G}_c(x,t) = \frac{t}{\alpha} T(x,t). \tag{4.1}$$

Equation (4.1) is given in [8].

(4) We comment now the numerical results obtained here. For (2.1) and (2.2), $\alpha, \beta \leqslant 1$, in the special case when $f_1(t) = f_2(t) = 0$ and on bounded domain, the results are presented in figure 2. It is observed from this figure that by decreasing β, the solution $T(x = \text{const}, t)$, in the beginning of the process, decreases more rapidly as β is decreased. However, as $t \to \infty$, the solution tends to zero more rapidly when β is larger.

(5) For the case $0 < \beta \leqslant \alpha \leqslant 2$, we treated first the signalling problem with $\alpha = 1.8, \beta = 0.9, \tau = 0.01, \mathcal{D} = 1$. The solution is shown in figure 3 for three different time instants. We solved the same problem with different sets of parameters. The numerical results show that for fixed α and t, the maximum of $T(x, t = \text{const})$ decreases with decreasing β.

(6) For the case $0 < \beta \leqslant \alpha \leqslant 2$, we treated also the Cauchy problem. The results are shown in figures 4–6. The values of the parameters were $\mathcal{D}=1, \tau = 0.001, \alpha = 2, t = 2$ and $\beta = 1.25, 1.5, 1.9$. We note that for the case $\beta = 2$, the classical wave equation is obtained and figures 4–6 represent an impulse in the form of Dirac distribution, travelling with the speed $v = \sqrt{\frac{\mathcal{D}}{1+\tau}}$. In our figures, it is seen that $\mathcal{G}_c(x,t)$ changes from

$$[\mathcal{G}_c(x,t)]^{\text{heat equation}} = \frac{1}{2\sqrt{\pi}} t^{-1/2} \exp(-x^2/(4t)) \tag{4.2}$$

corresponding to $\tau = 0, \beta = 1, \mathcal{D} = 1$ to

$$[\mathcal{G}_c(x,t)]^{\text{wave equation}} = \delta\left(\sqrt{\frac{\mathcal{D}}{1+\tau}} t - x\right) \tag{4.3}$$

corresponding to the wave equation.

In trying to reach as close as possible to solution (4.3), we solved (1.5) and (1.7) for $\mathcal{D}=(1+\tau), \alpha = \beta = 1.995$ and $t = 1$. We could not get convergence, for this set of parameters, in the series (3.13) and we used the integral form of the solution (3.18). The result is shown in figure 7. This figure shows the solution to (3.18) that is closest to (4.3).

Acknowledgments

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